MA 125 6D CALCULUS I

Test 4, November 18, 2015

Name (Print last name first): ....

Show all your work and justify your answer!

No partial credit will be given for the answer only!

PART I

You must simplify your answer when possible.

All problems in Part I are 8 points each.

1. If $f(x) = \ln(\tan(x))$, find the derivative $f'(x)$.

$$f'(x) = \left(\ln(\tan(x))\right)' = \frac{(\tan x)'}{\tan x} = \frac{\sec^2 x}{\tan x}.$$

2. Find the anti-derivative $F(x)$ of the function $f(x) = e^{-2x}$.

$$F(x) = \int e^{-2x}\,dx = -\frac{1}{2}e^{-2x} + C$$

3. Find the derivative of $f(x) = e^{\sin(x)}$.

$$f'(x) = (e^{\sin x})' = e^{\sin x}(\sin x)' = e^{\sin x} \cdot \cos x$$

4. Evaluate $\int \frac{x^2+1}{x^3+3x}\,dx$

$$u = x^3+3x \Rightarrow du = (3x^2+3)\,dx = 3(x^2+1)\,dx.$$

$$\Rightarrow \int \frac{x^2+1}{x^3+3x}\,dx = \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C$$

$$= \frac{1}{3}\ln|x^3+3x| + C$$

5. Solve $e^{5x+2} = 9$.

$$\ln e^{5x+2} = \ln 9$$

$$\ln e^{5x+2} = (5x+2)\ln e = 5x+2$$

$$\Rightarrow 5x = \ln 9 - 2$$

$$\Rightarrow x = \frac{\ln 9 - 2}{5}.$$

6. Solve $\ln(4x+3) = -2$.

$$\Leftrightarrow 4x+3 = e^{-2}$$

$$\Rightarrow 4x = e^{-2} - 3$$

$$\Rightarrow x = \frac{e^{-2}-3}{4}.$$

PART II

1. [**12 points**] Evaluate $\int_0^1 \frac{\sqrt{1+e^{-x}}}{e^x} dx$.

$$u = 1 + e^{-x} \Rightarrow du = -e^{-x} dx$$

$$\int \frac{\sqrt{1+e^{-x}}}{e^x} dx = -\int \sqrt{u}\, du = -\frac{2}{3} u^{\frac{3}{2}} + C$$

$$= -\frac{2}{3}(1+e^{-x})^{\frac{3}{2}} + C$$

$$\Rightarrow \int_0^1 \frac{\sqrt{1+e^{-x}}}{e^x} dx = -\frac{2}{3}(1+e^{-x})^{\frac{3}{2}} \Big]_0^1$$

$$= -\frac{2}{3}(1+e^{-1})^{\frac{3}{2}} + \frac{2}{3}(2)^{\frac{3}{2}}$$

7. Let $f(x) = x^3 - x + 1 = 0$. Compute the second approximate solution x_2, using Newton's method if the first approximate solution is $x_1 = -2$.

$$f'(x) = 3x^2 - 1 \Rightarrow f(-2) = (-2)^3 - (-2) + 1 = -5$$

$$f'(-2) = 3 \cdot (-2)^2 - 1 = 11$$

$$\Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -2 - \frac{f(-2)}{f'(-2)} = -2 - \frac{-5}{11}$$

$$= -1.545.$$

8. If Given $f(x) = x^3 + 5x + 1$ show first that $f(x)$ is one-to-one and next compute the derivative $(f^{-1})'(1)$

$$x_1 \neq x_2 \Rightarrow f(x_1) = x_1^3 + 5x_1 + 1$$

$$\neq.$$ one to one.

$$f(x_2) = x_2^3 + 5x_2 + 1$$

because $f(x_1) - f(x_2) = x_1^3 + 5x_1 + 1 - (x_2^3 + 5x_2 + 1)$

$$= x_1^3 - x_2^3 + 5(x_1 - x_2) = (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2 + 5) \neq 0.$$

$$\Rightarrow (f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} \underset{(1)}{=} \frac{1}{f'(0)} \underset{(2)}{=} \boxed{\frac{1}{5}.}$$

$$f^{-1}(1) = x \Leftrightarrow f(x) = 1 \Leftrightarrow x^3 + 5x + 1 = 1$$

$$\Leftrightarrow x^3 + 5x = 0 \Leftrightarrow x = 0. \quad \text{—(1)}$$

$$f'(x) = 3x^2 + 5 \Rightarrow f'(0) = 5 \quad \text{—(2)}$$

2. **[12 points]** Find the absolute Max and Min of the function $f(x) = x \ln x$ in the interval $[1, 2]$.

Step 1 $f(1) = 1 \ln 1 = 0.$

$f(2) = 2 \ln 2 \doteq 1.3863.$

Step 2 $f'(x) = (x \ln x)' = \cancel{\ln x + (x)' \ln x}$

$= (x)' \ln x + x (\ln x)'$

$= \ln x + x \cdot \frac{1}{x} = \ln x + 1 = 0$

$\Rightarrow \ln x = -1 \Rightarrow \boxed{x = e^{-1}}$

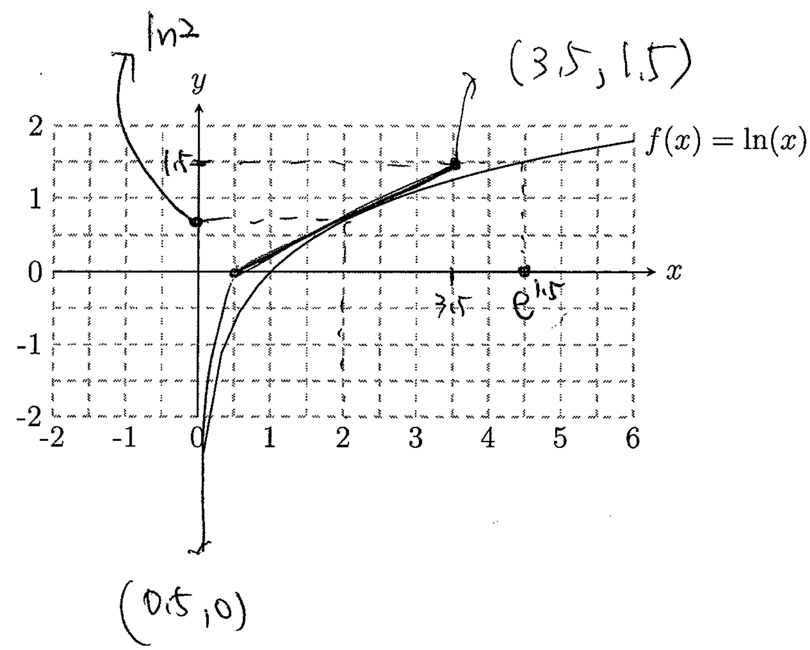
$\Rightarrow f(e^{-1}) = \cancel{x}\, e^{-1} \ln e^{-1} = -e^{-1} = -0.3679.$

Step 3 Candidates : 0, 1.3863, −0.3679

$\Rightarrow$ Abs max : 1.3863

Abs min : −0.3679.

3. [**12 points**] Given the graph of $y = \ln(x)$ below **read off**:
(1) the value $y = \ln(2)$.
(2) the value of $x = e^{1.5}$.
(3) Estimate the derivative $\ln'(2)$ (Hint: draw the tangent line and estimate its slope).
(4) Estimate the derivative of e^x at $x = 1.5$. **Indicate in the graph how you found your values; do not use your calculator to find these values!**



(3) $\ln'(2) = \dfrac{1.5 - 0}{3.5 - 0.5} = \dfrac{1.5}{3} = 0.5.$

(4) $f(x) = \ln x \Rightarrow f^{-1}(x) = e^x$ $\quad \boxed{f'(x) = \frac{1}{x}}$

$\Rightarrow$ ~~$e^{0.5}$~~ Derivative of e^x at $x = 1.5$

$= (f^{-1})'(1.5) = \dfrac{1}{f'(f^{-1}(1.5))} = \dfrac{1}{f'(e^{1.5})} = \dfrac{1}{\frac{1}{e^{1.5}}} = \boxed{e^{1.5}}$